NONHOMOGENEOUS POISSON FIELDS OF RANDOM LINES WITH APPLICATIONS TO TRAFFIC FLOW

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1. Introduction

This study was prompted by investigations of models of traffic flow on a highway through analyses of the structure and properties of Poisson fields of random lines in a plane. It is possible to view the trajectory of a car produced by its time and space coordinates on the highway as a straight line in that plane if the car travels at a constant speed once it enters the highway and then never leaves the highway. These traffic considerations plus the property of time invariance for traffic flow distributions lead to one model for traffic flow on a divided highway developed by Rényi [10]. This idealized model is simpler to study than the more realistic situation that provided Rényi's motivation and which he also subjects to analysis, namely, cars do lose time because of an overtaking of one car by another even on a divided highway with two lanes for traffic moving in one direction.

In his paper, Rényi found it convenient to start from the stochastic process of entrance times of the cars at a fixed point on the highway. Other authors start from the spatial process of cars distributed in locations along the highway at some fixed time according to some random law. The traffic flow results of Weiss and Herman [13] who study the spatial process for the idealized model are analogous to Rényi's results which stem from the temporal process. To demonstrate the equivalence of the two results, care must be taken to employ the appropriate measure in deriving distributions related to traffic flow. Both Rényi, and Weiss and Herman, achieved asymptotic results for traffic flow distributions. We will reproduce both results in Section 4 as special cases of our development of traffic flow models through the structure of random lines in the plane. It should be mentioned here that Brown [3] reconsiders Rényi's idealized model and derives exact distributions rather than asymptotic distributions for spatial and speed distributions of cars.

To this point we have not been specific about the random processes governing entrance times of cars on a highway, positions of cars on a highway, and speed distributions of cars on the highway. The Poisson process is the assumed machinery governing car entrance times or equivalently car positions and the speed distributions for each car are assumed to be identically and independently distributed (i.i.d.). Starting from the spatial process, Breiman [2] considered the idealized model and proved that the Poisson process is the only process obeying the time invariance property—namely if at a time t_0, the spatial process is Poisson with specific parameter, and the speeds of the cars are i.i.d. with respect to each other and the positions of the cars at time t_0, then the process will have the same properties at any other time t.

Obviously, other results for Poisson processes can be germane to traffic flow situations and similarly this can be so for results in queueing processes. There is a vast literature in both subjects. However, it is pertinent to this exposition to mention some results for the $M/G/\infty$ queue. The highway can be regarded as the infinite server for each car suffers no delay when it enters the highway in our model; in addition the input is Poisson and the service time distribution is the distribution of the distances traveled by each car before it overtakes or is overtaken by another car on the highway or equivalently the distribution of the time expended until an overtaking occurs. In a paper on Markov processes, Kingman [6] arrives at a general formulation that can be reduced to our idealized traffic flow model or equivalently the $M/G/\infty$ queue. This produces the result that the distribution of cars on the highway is Poisson with parameter

$$(1.1) \qquad \omega \int_0^\infty \frac{1}{v} \, dG(v).$$

where ω is the parameter of the Poisson process for cars entering the highway and $G(v)$ is the cumulative distribution function for the speed of each car. This result is employed by Rényi in his paper where he cites other authors who have produced it. It also falls out of the development in this paper and appears in Theorem 4.2.

Along these lines there is a recent paper by Brown [4] in which he discusses an estimation procedure for G in the $M/G/\infty$ queue for which data are kept only on the times cars enter or leave the highway without identification of cars (that is, no pairing of entering and departing times for any one car). This is a different model from the one to which we give central attention in this paper. In its representation in the time-space plane we would have one straight line going through an origin on the time axis or equivalently on the spatial axis to indicate an arbitrary car (or observer car) always traveling on the highway at some constant speed, v_0, but all the other cars would be indicated by line segments from whose lengths we could get distance traveled on highway (still assumed to be i.i.d.) and from whose orientation angle with the t axis we could

get the speed of the car. This produces the problem of the distribution of the number of intersections made by the line segments with the fixed line, an interesting but unsolved problem in geometrical probability.

However, this serves to return us to the central issue of intersections of random lines in the plane and its relationship to traffic flow models. We now turn to the formulation where the arbitrary car and all other cars are indicated by straight lines in the plane. The number of intersections of the arbitrary line (observer car) by the other lines determines the number of overtakings of slower cars made by the observer car plus the number of times it was overtaken by faster cars. We are interested in this distribution and also in the distributions of faster car overtakings of the observer car and the overtaking of slower cars by the observer car.

The structure of random lines in the plane and the properties resulting from a specific structure are therefore pertinent to analyses of traffic flow for our idealized model. The notion of a homogeneous Poisson field of random lines in the plane and its consequences have been developed by Miles in several papers [7], [8]. Additional development for nonhomogeneous Poisson fields of random lines is required for study of traffic flow models. In subsequent sections, we provide a formulation for a nonhomogeneous Poisson field of random lines and develop its structure and characteristics. This makes it possible to provide a different proof of Rênyi's theorem and the Weiss and Herman result on traffic flow and allows for further understanding of traffic flow models. It also provides a format for viewing their results as special cases of a more general model. In fact, this model provides a unified treatment for viewing any aspect of the idealized traffic flow model.

2. Development

First we formalize the notion of straight lines distributed "at random" throughout the plane. We will describe the plane in terms of (t, x) coordinates for subsequently the t axis will be employed to register time of arrival of cars at a fixed point on a highway and the x axis will in similar fashion report on spatial positions of cars on a highway at a fixed point in time. Naturally the time invariance property will insure that the conditions will prevail at any point in time. Any line in the (t, x) plane can be represented as

$$(2.1) \qquad p = t \cos \alpha + x \sin \alpha, \qquad -\infty < p < \infty, 0 \leqq \alpha < \pi,$$

where p is the signed length of the perpendicular to the line from an arbitrary origin 0, and α is the angle this perpendicular makes with the t axis (Figure 1). Note that if the intersection of the perpendicular with the line is in the third or fourth quadrant, p is taken to be negative. A set of lines $\{(p_i, \alpha_i): i = 0, \pm 1, \pm 2, \cdots\}$ constitutes a Poisson field under the following conditions.

(1) The distances $\cdots \leqq p_{-2} \leqq p_{-1} \leqq p_0 \leqq p_1 \leqq p_2 \leqq \cdots$ of the lines from an arbitrary origin 0, arranged according to magnitude represent the coordinates of the events of a Poisson process with constant parameter, say λ. Thus, the

number of p_i in an interval of length L has a Poisson distribution with mean λL.

(2) The orientations α_i of each line with a fixed but arbitrary axis (say the t axis) in the plane are independent and obey a uniform distribution in the interval $[0, \pi)$.

Thus, a reasonable representation of random lines in the plane is that of the Poisson field. This definition of randomness for lines in the plane also has the property that the randomness is unaffected by the choice of origin or line to serve as t axis, since it can be demonstrated that except for a constant factor $\int dp\, d\alpha$ is the only invariant measure under the group of rotations and translations that transform the line (p, α) to the line (p', α'). We will return to this structure and its characteristics, but now we employ it as a point of departure to initiate discussion of a nonhomogeneous Poisson field of random lines. To achieve this we will relax condition (2) above and ask only that the α_i be identically and independently distributed (i.i.d.).

For ease in the algebra of our traffic flow models, we will employ instead of α_i an angle formed by the intersection of the t axis with a line in the plane and we label this θ (Figure 1); note that $v = \tan \theta$. Also we will only be concerned with
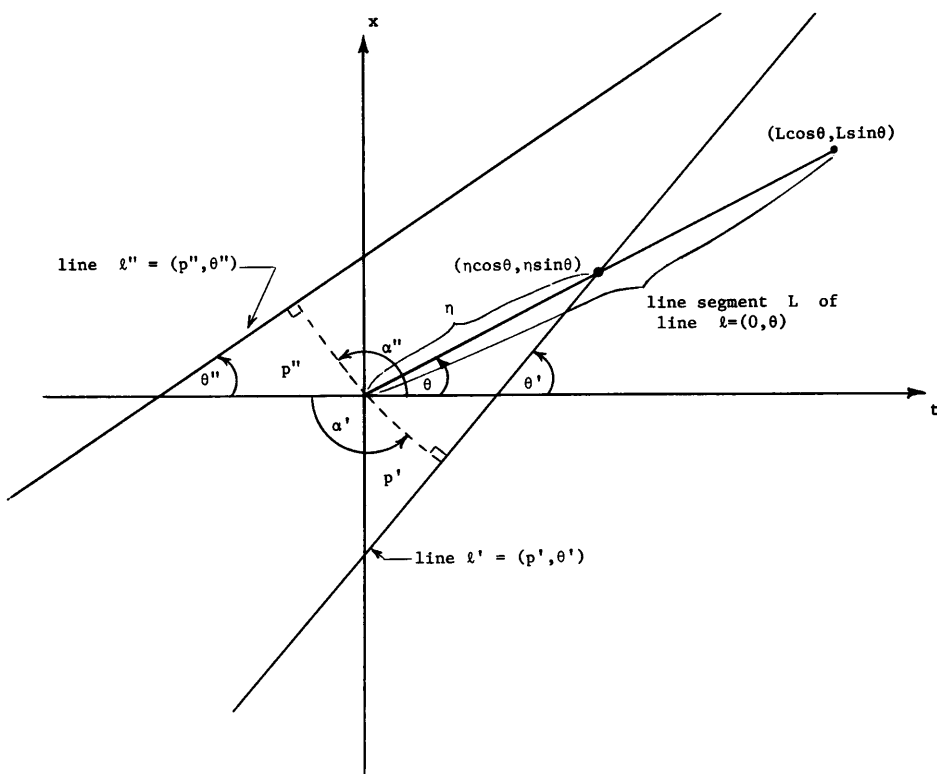

FIGURE 1

those lines where p_i falls in the second or fourth quadrant since this will yield all positive car velocities. The inclusion of the p_i in the first and third quadrant does not complicate the mathematical development, but they are not relevant. Thus, $\alpha = \frac{1}{2}\pi + \theta$ and the lines of interest will now be parametrized by (p, θ) where

$$(2.2) \qquad\qquad p = -t \sin \theta + x \cos \theta, \qquad\qquad 0 \leq \theta < \frac{1}{2}\pi.$$

Equation (2.2) takes care of the sign of p for it insures that p will be positive if it is in the second quadrant and negative in the fourth quadrant.

The set $\mathscr{L}$ of lines $\{(p_i, \alpha_i): i = 0, \pm 1, \pm 2, \cdots\}$ becomes a nonhomogeneous Poisson field if we require invariant measure only under translation, and we look into this situation because it will be helpful in our traffic flow models. Under this constraint, we now have the same conditions except that the orientation angles α_i of each line are i.i.d. random variables with common distribution function in the interval $[0, \pi)$. Thus, $\int dp d\alpha$ is no longer the appropriate measure. The diagram in Figure 1 delineates the situation where the origin can be arbitrarily chosen at any point on a specific and fixed t axis because invariance is preserved now only under translation.

The orientations θ_i are independent and identically distributed with common distribution F in the interval $[0, \frac{1}{2}\pi)$ and further the sequence of values $\langle \theta_i \rangle$ are independent of $\langle p_i \rangle$. This is equivalent to the statement that the velocities of cars, namely, $v_i = \tan \theta_i$ are independent and identically distributed with common distribution G on $[0, \infty)$ and thus $\langle v_i = \tan \theta_i \rangle$ are independent of $\langle p_i \rangle$.

When $\theta_0 = 0, p_0 = 0$, the traffic flow is characterized by a distribution of time intercepts on the t axis; when $\theta_0 = \frac{1}{2}\pi, p_0 = 0$, the traffic flow is characterized by a distribution of cars spaced along the x axis. For any other value of θ, the traffic flow is measured along a trajectory line. In the traffic literature, trajectories for low density traffic flow (no delays in overtaking) may be assumed to be linear in the time-space plane. Thus in any development, we must employ the appropriate measure to characterize distributions of traffic flow in such matters, for example, as distribution of number of overtakings. For our purposes where Poisson processes will be the underpinning for traffic flow in both spatial and temporal processes, the evaluation of the appropriate Poisson intensity parameter will be paramount as will be the relationships between these parameters for different measures.

The following exposition and the diagram in Figure 2 are included to make clear how the departure to nonhomogeneous Poisson fields occurs. Consider the (p, θ) plane. The homogeneous Poisson field occurs when points on the p axis follow a Poisson process with parameter λ independent of p and θ is uniformly distributed from 0 to $\frac{1}{2}\pi$. Given a fixed interval on p containing exactly n points, each follows a uniform distribution whose range is the length of the interval. When the interval is of unit length the density is dp. Similarly the density for θ is $d\theta$ and the joint density is $dp d\theta$ leading to $\int dp d\theta$ as the measure. This is invariant under rotations and translations. If θ_i is i.i.d. but not uniformly
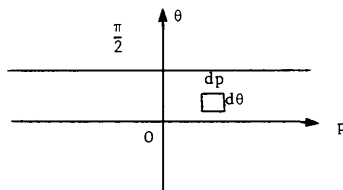
FIGURE 2

distributed, then the Poisson process for points of intersection along any trajectory line (p, θ) is maintained but the density for θ is no longer $d\theta$. Thus, any $dp\,d\theta$ rectangle as in Figure 2 will have the same measure only under translation on the p axis.

Also if θ is uniformly distributed but the points on the p axis follow a Poisson with parameter $\lambda(p)$, we obtain a nonhomogeneous Poisson field of random lines. If there are departures in the structure of both θ_i and p_i as listed above, then we obviously have a nonhomogeneous Poisson field of random lines where a Poisson process for points of intersections along any trajectory line (p, θ) will be maintained but the counting will be measured by the values of (p, θ) or equivalently the values of (t, x).

3. Basic results

The main results for nonhomogeneous Poisson fields of random lines $\mathscr{L}$ are discussed in this section. All sets under investigation are assumed to be measurable and events of probability zero are neglected. For instance, a possible realization of $\mathscr{L}$ is one in which there are no lines at all with probability zero and this is omitted. Many of the following results must be qualified by the phrase "with probability one"; however, this is often omitted for brevity. One basic feature of a special nonhomogeneous Poisson field of random lines, where p is Poisson with parameter λ and θ is i.i.d. is given in the following theorem.

THEOREM 3.1. *Points of intersections of such random lines $\mathscr{L}$ and any arbitrary and fixed line (p_0, θ_0) form a Poisson process with parameter $\lambda(\theta_0)$, where*

$$(3.1) \qquad \lambda(\theta_0) = \lambda \cos \theta_0 \int_0^\infty |v - \tan \theta_0| (1 + v)^{-1/2} \, dG(v)$$

and λ is the parameter of the Poisson field of random lines and $v = \tan \theta$.

NOTE.   The counting is done on the arbitrary and fixed line (p_0, θ_0) and of course $\langle v = \tan \theta \rangle$ for all the lines in $\mathscr{L}$. In our traffic model, a point of intersection on the line (p_0, θ_0) when represented in the coordinates of the time-space plane (t_0, x_0) may be viewed in the following traffic sense—t_0 is the actual time of car overtaking and x_0 is the actual spatial position where the car overtaking event occurs. This is developed more fully in Section 4.

PROOF. We note that the random mechanism in $\mathscr{L}$ is invariant under translation and is thus unaffected by the choice of origin; hence, we can have the arbitrary line go through the origin such that the segment of line $(0, \theta_0)$ with length L emanates from the origin and θ_0 is the fixed angle associated with this arbitrary line. Now denote η as the length of this segment measured from the origin to the point of intersection with another line $\ell' = (p', \theta') \in \mathscr{L}$ (see Figure 1). We can classify lines in $\mathscr{L}$ that intersect with line segment L into two groups, namely:

(1) for θ such that $0 < \theta < \theta_0$, $p > 0$ we have $\eta \sin (\theta_0 - \theta) = p$, if and only if

$$(3.2) \qquad 0 < \eta = p \csc (\theta_0 - \theta) < L.$$

(2) for θ such that $\theta_0 < \theta < \frac{1}{2}\pi$, $p < 0$ we have $\eta \sin (\theta - \theta_0) = -p$, if and only if

$$(3.3) \qquad 0 < \eta = -p \csc (\theta - \theta_0) < L.$$

Let N_L denote the number of lines in $\mathscr{L}$ intersecting the segment of length L. Then we will show that

$$(3.4) \qquad Pr \{N_L = n\} = \exp \{-L\lambda\mu\} \frac{(L\lambda\mu)^n}{n!},$$

and upon evaluation, that

$$(3.5) \qquad \mu = \cos \theta_0 \int_0^\infty |v - \tan \theta_0| (1 + v^2)^{-1/2} \, dG(v).$$

Recall that $\cos \theta_0$ and $\tan \theta_0$ are constants depending on the θ_0 of the arbitrary line. Denote N_p the number of random lines whose signed distance p, to the origin is between $-L \sin \theta_0$ and $L \cos \theta_0$. Then

$$(3.6) \qquad Pr \{N_L = n\} = \sum_{m=0}^\infty Pr \{N_L = n \mid N_p = m\} Pr \{N_p = m\}.$$

Clearly, no line can intersect the segment L unless its minimum distance to the origin is between $-L \sin \theta_0$ and $L \cos \theta_0$. Thus, N_p must be more than N_L; that is,

$$(3.7) \qquad Pr \{N_L = n \mid N_p = m\} = 0 \quad \text{for } n > m,$$

and therefore

$$(3.8) \qquad Pr \{N_L = n\} = \sum_{m=n}^\infty Pr \{N_L = n \mid N_p = m\} Pr \{N_p = m\}.$$

Let $\mu = Pr \{N_L = 1 \mid N_p = 1\}$. Then since the random lines are independent, that is, sequences $\langle v_i \rangle$ and $\langle p_i \rangle$ are independent, we have

$$(3.9) \qquad Pr \{N_L = n \mid N_p = m\} = \binom{m}{n} \mu^n (1 - \mu)^{m-n} \quad \text{for } m \geqq n.$$

By the definition of random lines $\mathscr{L}$, we have

$$(3.10) \quad Pr\{N_p = m\} = \exp\{-\lambda L(\sin\theta_0 + \cos\theta_0)\} \frac{[\lambda L(\cos\theta_0 + \sin\theta_0)]^m}{m!}.$$

Thus,

$$(3.11) \quad Pr\{N_L = n\} = \sum_{m=n}^{\infty} \binom{m}{n} \mu^n (1-\mu)^{m-n}$$

$$\exp\{-\lambda L(\sin\theta_0 + \cos\theta_0)\} \frac{[\lambda L(\cos\theta_0 + \sin\theta_0)]^m}{m!}$$

$$= \exp\{-\lambda L\mu(\sin\theta_0 + \cos\theta_0)\} \frac{[\lambda L\mu(\sin\theta_0 + \cos\theta_0)]^n}{n!}.$$

Now we evaluate μ, the probability that a line whose minimum signed distance to the origin is between $-L\sin\theta_0$ and $L\cos\theta_0$, will intersect the segment L. Write

$$(3.12) \quad \mu = Pr\{N_L = 1 \,|\, N_p = 1\}$$

$$= Pr\{0 < \eta < L \,|\, -L\sin\theta_0 < p < L\cos\theta_0\}$$

$$= Pr\{0 < \eta < L; 0 < \theta < \theta_0 \,|\, -L\sin\theta_0 < p < L\cos\theta_0\}$$

$$+ Pr\{0 < \eta < L; \theta_0 < \theta < \tfrac{1}{2}\pi \,|\, -L\sin\theta_0 < p < L\cos\theta_0\}$$

$$= \frac{1}{L(\sin\theta_0 + \cos\theta_0)} \left[\int_{-L\sin\theta_0}^{0} Pr\{0 < \eta < L, 0 < \theta < \theta_0 \,|\, p\}\, dp \right.$$

$$\left. + \int_{0}^{L\cos\theta_0} Pr\{0 < \eta < L; \theta_0 < \theta < \tfrac{1}{2}\pi \,|\, p\}\, dp \right].$$

Therefore, we have

$$(3.13) \quad L\mu(\cos\theta_0 + \sin\theta_0) = \int_{-L\sin\theta_0}^{0} Pr\{0 < \eta < L; 0 < \theta < \theta_0 \,|\, p\}\, dp$$

$$+ \int_{0}^{L\cos\theta_0} Pr\{0 < \eta < L; \theta_0 < \theta < \tfrac{1}{2}\pi \,|\, p\}\, dp$$

$$= L\cos\theta_0 \left[\int_{0}^{\tan\theta_0} (\tan\theta_0 - v)(1 + v^2)^{-1/2}\, dG(v) \right.$$

$$\left. + \int_{\tan\theta_0}^{\infty} (v - \tan\theta_0)(1 + v^2)^{-1/2}\, dG(v) \right].$$

Hence, we may conclude that

$$(3.14) \quad \mu = \frac{\cos\theta_0}{\cos\theta_0 + \sin\theta_0} \int_{0}^{\infty} |v - \tan\theta_0|(1 + v^2)^{-1/2}\, dG(v),$$

and this in turn gives the result that

$$(3.15) \quad Pr\{N_L = n\} = \exp\left\{-\lambda L \cos\theta_0 \int_0^\infty |v - \tan\theta_0|(1 + v^2)^{-1/2} \, dG(v)\right\}$$

$$\cdot \left[\lambda L \cos\theta_0 \int_0^\infty |v - \tan\theta_0|(1 + v^2)^{-1/2} \, dG(v)\right]^n (n!)^{-1}.$$

That is, N_L, the number of intersections with segment L from lines in $\mathscr{L}$, follows the Poisson distribution with parameter

$$(3.16) \qquad \lambda \cos\theta_0 \int_0^\infty |v - \tan\theta_0|(1 + v^2)^{-1/2} \, dG(v).$$

Let us denote N_L^+ the number of lines in $\mathscr{L}$ intersecting the segment L with $\theta_0 < \theta < \frac{1}{2}\pi$ and denote N_L^- the number of lines in $\mathscr{L}$ intersecting the segment L with $0 < \theta < \theta_0$. (Clearly, $N_L = N_L^+ + N_L^-$.) Theorem 3.1 permits us to state the following result immediately.

THEOREM 3.2. *Let $\langle \tau_i^- \rangle$ be points of intersections of random lines $\mathscr{L}$ with $\theta_0 < \theta < \frac{1}{2}\pi$ and any arbitrary line (p_0, θ_0) and let $\langle \tau_i^+ \rangle$ be points of intersections of random lines $\mathscr{L}$ with $0 < \theta < \theta_0$ and line (p_0, θ_0). Then $\langle \tau_i^+ \rangle$ and $\langle \tau_i^- \rangle$ form two independent Poisson processes, with parameters*

$$(3.17) \qquad \lambda^+(\theta_0) = \lambda \cos\theta_0 \int_0^{\tan\theta_0} (\tan\theta_0 - v)(1 + v^2)^{-1/2} \, dG(v)$$

and

$$(3.18) \qquad \lambda^-(\theta_0) = \lambda \cos\theta_0 \int_{\tan\theta_0}^\infty (v - \tan\theta_0)(1 + v^2)^{-1/2} \, dG(v).$$

In traffic terms, $\lambda^+(\theta_0)$ is the intensity of the Poisson process generated by the fixed car $K(p_0, \theta_0)$ overtaking slower cars and $\lambda^-(\theta_0)$ for faster cars overtaking $K(p_0, \theta_0)$. We remark here that if $\theta_0 = 0$, $p_0 = 0$, namely the t axis, then $\lambda^+(0) = 0$ and $\lambda^-(0) = \lambda \int_0^\infty v(1 + v^2)^{-1/2} \, dG(v)$, that is, if the random variable N_p is distributed according to the Poisson distribution with parameter λ, then the points of intersection of random lines $\mathscr{L}$ with the t axis form a point process distributed according to the Poisson distribution with parameter $\lambda \int_0^\infty v(1 + v^2)^{-1/2} \, dG(v)$. Similarly, if $\theta_0 = \frac{1}{2}\pi$, $p_0 = 0$, namely the x axis, then

$$(3.19) \qquad \lambda(\tfrac{1}{2}\pi) = \lambda \int_0^\infty (1 + v^2)^{-1/2} \, dG(v),$$

and the counting is done along the spatial axis. The corresponding $\lambda^+(\frac{1}{2}\pi) = \lambda(\frac{1}{2}\pi)$ and $\lambda^-(\frac{1}{2}\pi) = 0$. Hence, we have established the following result.

THEOREM 3.3. (i) *Points of intersection of the field $\mathscr{L}$ and the t axis (temporal counting) form a Poisson process with parameter λ_t where*

$$(3.20) \qquad \lambda_t = \lambda \int_0^\infty v(1 + v^2)^{-1/2} \, dG(v).$$

(ii) *Points of intersection of the field $\mathscr{L}$ and the x axis (spatial counting) form a Poisson process with parameter λ_x, where*

$$(3.21) \qquad \lambda_x = \lambda \int_0^\infty (1 + v^2)^{-1/2} \, dG(v).$$

Let the sequence $\langle \tau_{1k} \rangle$ denote the instants when the arbitrary line (p_0, θ_0) segment of length L intersects random lines of $\mathscr{L}$ whose orientations θ_i belong to a given set Θ_1 and sequence $\langle \tau_{2k} \rangle$ denote the instants when the arbitrary line (p_0, θ_0) segment of length L intersects random lines of $\mathscr{L}$ whose orientations θ_i belong to a given set Θ_2. We now state a generalized version of the results in Theorem 3.4.

THEOREM 3.4.   *If $\Theta_1 \cap \Theta_2 = \varnothing$, then the two sequences $\langle \tau_{1k} \rangle$ and $\langle \tau_{2k} \rangle$ form two independent Poisson processes with parameters $\lambda_1(\theta_0)$ and $\lambda_2(\theta_0)$, respectively, where*

$$(3.22) \qquad \lambda_i(\theta_0) = \lambda \int_{\tan \Theta_i} |v - \tan \theta_0| (1 + v^2)^{-1/2} \, dG(v), \qquad i = 1, 2,$$

and

$$(3.23) \qquad \tan \Theta_i = \{ v \,|\, v = \tan \theta \text{ such that } \theta \in \Theta_i \}. \qquad i = 1, 2.$$

The details of the proof are omitted since it is essentially the proof used in Theorem 3.1.

In the next paragraphs, we establish some similar results employing $\langle \tau_i \rangle$ and $\langle \eta_i \rangle$, where the τ_i are the arrival times of cars on a highway measured from a fixed position, say $x = 0$, and η_i are corresponding positions of these cars on the highway at a fixed time, say t_i. Let us denote $\langle \tau_i \rangle$ the sequence of points of intersections of a given random family of lines $\mathscr{A}$ with the t axis. Let $\langle v_i = \tan \theta_i \rangle$ be the sequence of i.i.d. random variables. Those θ_i are the orientations of random lines in $\mathscr{A}$. Denote $\langle \eta_i \rangle$ the sequence of points of intersections of random lines in $\mathscr{A}$ with the x axis. We employ $\mathscr{A}$ instead of $\mathscr{L}$ for now we do not wish to assume as in $\mathscr{L}$ that $\langle p_i \rangle$ and $\langle v_i \rangle$ are independent sequences. In the following theorems, we will employ $\langle \eta_i \rangle$ and $\langle v_i \rangle$, or $\langle \tau_i \rangle$ and $\langle v_i \rangle$ as independent sequences and these assumptions will be made specific in each statement of the theorem. The results can be stated as follows.

THEOREM 3.5.   *If $\langle \tau_i \rangle$ forms a Poisson process with parameter λ_τ and sequences $\langle \tau_i \rangle$ and $\langle v_i \rangle$ are independent, then $\langle p_i \rangle$ forms a Poisson process with parameter $\lambda_\eta \int_0^\infty [(1 + v^2)^{1/2}/v] \, dG(v)$.*

Similarly, we have:

THEOREM 3.6.   *If $\langle \eta_i \rangle$ forms a Poisson process with parameter λ_η and sequence $\langle \eta_i \rangle$ and $\langle v_i \rangle$ are independent, then $\langle p_i \rangle$ forms a Poisson process with parameter $\lambda_\eta \int_0^\infty (1 + v^2)^{1/2} \, dG(v)$.*

We shall give a proof of Theorem 3.5. The proof of Theorem 3.6 is similar and hence is omitted.

PROOF OF THEOREM 3.5.   Based on the proof similar to that used in Theorem 3.1, we denote $N_t(c)$ the number of lines in $\mathscr{A}$ intersecting the t axis in an interval of length c and denote N_p for the number of lines in $\mathscr{A}$ whose p_i is bounded by $0 < p_i < p$. Then it is clear we want to show that

$$(3.24) \quad Pr\{N_p = n\}$$
$$= \exp\left\{-\lambda p \int_0^\infty \frac{(1 + v^2)^{1/2}}{v}\, dG(v)\right\}\left[\lambda p \int_0^\infty \frac{(1 + v^2)^{1/2}}{v}\, dG(v)\right]^n (n!)^{-1}.$$

Following the previous development, we arrive at

$$(3.25) \qquad Pr\{N_p = n\} = \lim_{c \to \infty} \exp\{-\lambda_1 C\mu_c\} \frac{(\lambda_1 C\mu_c)^n}{n!}$$

and

$$(3.26) \qquad \mu_c = Pr\{N_p = 1 \,|\, N_t(c) = 1\}.$$

It remains to show that

$$(3.27) \qquad \lim_{c \to \infty} c\mu_c = p \int_0^\infty \frac{(1 + v^2)^{1/2}}{v}\, dG(v),$$

$$(3.28) \qquad \mu_c = Pr\{N_p = 1 \,|\, N_t(c) = 1\} = \frac{1}{c}\int_0^c Pr\{N_p = 1 \,|\, \tau\}\, d\tau,$$

$$(3.29) \qquad c\mu_c = \int_0^c Pr\{N_p = 1 \,|\, \tau\}\, d\tau,$$

$$(3.30) \qquad \lim_{c \to \infty} c\mu_c = \int_0^\infty Pr\{N_p = 1 \,|\, \tau\}\, d\tau$$
$$= \int_0^\infty Pr\left\{0 < \theta < \sin^{-1}\frac{p}{\tau}\,\middle|\,\tau\right\}\, d\tau$$
$$= \int_0^\infty \int_0^{p/\sin\tan^{-1}v}\, d\tau\, dG(v)$$
$$= p \int_0^\infty \frac{(1 + v^2)^{1/2}}{v}\, dG(v).$$

This completes the proof of Theorem 3.5.

4. Traffic flow applications

The purpose of this section is to discuss a number of results that can be related to low density traffic flow models on an infinite highway in the light of the developments in Section 3. These models were initiated and developed principally in papers by Rényi [10]. Weiss and Herman [13], and Breiman [2], sometimes

without specific reference to low density traffic flow. Of the theorems presented in this section, some are known but all the proofs are new and developed in a unified manner.

Rényi [10] has developed and analyzed a model of traffic flow on a divided highway that extends in one direction out to infinity without traffic lights or other barriers. It is assumed that the speed of each car is constant but its value is governed by a random variable and passing is always achieved without delays. Assuming that the temporal distribution of cars is described by a Poisson process, Rényi obtained some asymptotic results for the spatial distribution of cars along the highway. In what follows, we shall reproduce Rényi's theorems and include other results dealing with low density traffic flow. It will also be demonstrated for Rényi's model, that if the spatial distribution of cars is assumed to obey a Poisson process then the temporal distribution of cars (that is, arrival times at some fixed position) is again a Poisson process. This new result establishes a crucial structural property of Rényi's model for low density traffic. In detail, the assumptions of Rényi's model are:

(a) instants $\langle t_i \rangle_{i=1}^{\infty}$ at which cars enter the highway at a fixed position form a homogeneous Poisson process with parameter ω;

(b) a car arriving at a certain point on the highway at instant t_i chooses a velocity V_i and then moves with this constant velocity; the random variables $\langle V_k \rangle$ are independently and identically distributed with distribution function $G(v) = Pr\{V \leq v\}$ and sequences $\{V_k\}$ and $\langle t_k \rangle$ are independent;

(c) $\int_0^{\infty} (1/v) \, dG(v) < \infty$, that is, the mean value of $1/v$ is finite; without this condition a traffic jam would arise and make all traffic flow impossible;

(d) no delay in overtaking a car traveling at a slower speed when it is approached.

Suppose an arbitrary car $K(t_0, v_0)$ arrives at some fixed point of the highway at time t_0, where it assumes and maintains the fixed velocity v_0. Let $\langle t_k^+ \rangle$ denote the instants at which the car $K(t_0, v_0)$ overtakes slower cars and $\langle t_k^- \rangle$ denote instants at which car $K(t_0, v_0)$ is overtaken by faster cars. Rényi has obtained the following asymptotic result.

THEOREM 4.1. (Rényi).   *The instants $\{t_k^+\}$ and $\{t_k^-\}$ form two independent homogeneous Poisson processes, with parameters:*

$$(4.1) \qquad \omega^+(v_0) = \omega \int_0^{v_0} \frac{v_0 - v}{v} \, dG(v), \qquad \omega^-(v_0) = \omega \int_{v_0}^{\infty} \frac{v - v_0}{v} \, dG(v).$$

Rényi's proof of the above theorem is based on the following two properties of Poisson processes:

(A) if $\langle t_i \rangle$ are the instants of time when an event occurs in a homogeneous Poisson process with parameter ω, and $\zeta_1, \zeta_2, \cdots$, is a sequence of independent positive random variables, each having the same distribution $G(\zeta)$ and each is independent of the process $\{t_k\}$, then the time instants $t_k \zeta_k, k = 1, 2, \cdots$, also form a homogeneous Poisson process with density

$$(4.2) \qquad \omega^* = \omega \int_0^\infty \frac{1}{\zeta} \, dG(\zeta);$$

(B) if a subsequence $\{t_{v_k}\}$ of the instants $\{t_k\}$, in which an event occurs in a Poisson process with density ω, is selected at random in such a way that for each j the probability of the event A_j that j should belong to the subsequence $\{v_k\}$ is equal to r, $0 < r < 1$, and the events A_j, $j = 1, 2, \cdots$, are independent, and if $\{t_{u_k}\}$ are the instants that are not selected, (that is, j belongs to the sequence $\{u_k\}$ if and only if it does not belong to the sequence $\{v_k\}$), then $\{t_{v_j}\}$ and $\{t_{u_k}\}$ are two independent Poisson processes with density ωr and $\omega(1 - r)$.

It is now known from a result of Wang [12] that property (B), in some sense, is a characteristic property for Poisson processes. We shall establish Theorem 4.1 without using property (A). Thus, it may be inferred that property (B) implies property (A).

PROOF OF THEOREM 4.1.   The trajectory of any car in the time-space diagram in the preceding section for Rényi's low density traffic model is realized by a straight line. Let us denote the trajectories of all cars on the highway as a set $\mathscr{A}$. Then it is clear that $\mathscr{A}$ possesses the properties of a nonhomogeneous Poisson field of random lines. Denote M_L^+ the number of lines in $\mathscr{A}$ that intersect segment L from below and M_p the number of lines in $\mathscr{A}$ whose arrival times are in $(0, t_0)$. Then

$$(4.3) \qquad Pr\{M_L^+ = n\} = \sum_{m=n}^\infty \binom{m}{n} \mu^n (1 - \mu)^{m-n} \exp\{-\omega t_0\} \frac{(\omega t_o)^m}{m!}$$

$$= \exp\{-\omega t_0 \mu\} \frac{(\omega t_0 \mu)^n}{n!},$$

where

$$(4.4) \qquad \mu = Pr\{M_L^+ = 1 \,|\, M_p = 1\}$$

$$= Pr\left\{\theta \geqq \tan^{-1} \frac{x_0}{t_0 - p} \,\middle|\, 0 < p < t_0\right\}$$

$$= \frac{1}{t_0} \int_0^{t_0} Pr\left\{V \geqq \frac{x_0}{t_0 - p}\right\} dp$$

$$= \int_{v_0}^\infty \frac{v - v_0}{v} \, dG(v),$$

and $v_0 = x_0/t_0$.

Similarly, we define M_L^- as the number of lines in $\mathscr{A}$ intersecting L from above and M_p^c as the number of lines in $\mathscr{A}$ whose arrival times fall in the interval $-c$ and 0, $c > 0$. We can compute

$$(4.5) \qquad Pr\{M_L^- = n\} = \lim_{c \to \infty} \exp\{-\omega c \mu_c^*\} \frac{(\omega c \mu_c^*)^n}{n!}$$

where $\mu_c^* = Pr\{M_L^- = 1 \mid M_p^c = 1\}$ and

(4.6)
$$\lim_{c \to \infty} c\mu_c^* = \lim_{c \to \infty} cPr\{M_L^- = 1 \mid M_p^c = 1\}$$

$$= \lim \int_{-c}^0 \int_0^{x_0/(t_0-p)} dG(v)\, dp$$

$$= \int_{-\infty}^0 \int_0^{x_0/(t_0-p)} dG(v)\, dp$$

$$= \int_0^{v_0} \frac{x_0 - vt_0}{v}\, dG(v)$$

$$= t_0 \int_0^{v_0} \frac{v_0 - v}{v}\, dG(v).$$

Random variables M_L^- and M_L^+ are independent because the events involved come from disjoint intervals. This completes the proof of Theorem 4.1.

The counting interval employed in the above theorem is on the time axis. In what follows, a similar approach to the problem dealing with a spatial counting interval is employed and produces some interesting results.

Denote M^- the number of lines in $\mathscr{A}$ intersecting L from above and M^+ the number of lines from below. Denote M_{x_0} the number of lines in $\mathscr{A}$ whose spatial positions at $t = 0$ are between 0 and x_0 and similarly $M_{x_0}^c$ for 0 and $-c$. Let λ^* be the spatial density. Then we have

(4.7)
$$Pr\{M^- = n\} = \exp\{-\lambda^* x_0 \mu\} \frac{(\lambda^* x_0 \mu)^n}{n!}.$$

where

(4.8)
$$\mu = Pr\{M^- = 1 \mid M_{x_0} = 1\}$$

$$= \frac{t_0}{x_0} \int_0^{v_0} (v_0 - v)\, dG(v).$$

and

(4.9)
$$Pr\{M^+ = n\} = \lim_{c \to \infty} \sum_{m=n}^{\infty} Pr\{M^+ = n \mid M_{x_0}^c = m\} Pr\{M_{x_0}^c = m\}$$

$$= \lim_{c \to \infty} \exp\{-\lambda^* c_1^c\} \frac{(\lambda^* c \mu_1^c)^n}{n!}$$

where $\mu_1^c = Pr\{M^+ = 1 \mid M_{x_0}^c = 1\}$. It can be easily verified that

(4.10)
$$\lim_{c \to \infty} c\mu_1^c = t_0 \int_{v_0}^{\infty} (v - v_0)\, dG(v)$$

and random variables M^- and M^+ are independent. Now denote $M = M^+ + M^-$. We conclude that

$$(4.11) \quad Pr\left\{M = n\right\} = \exp\left\{-\lambda^* t_0 \int_0^\infty |v_0 - v|\, dG(v)\right\}$$

$$\cdot \left[\lambda^* t_0 \int_0^\infty |v_0 - v|\, dG(v)\right]^n (n!)^{-1}.$$

The above result appeared initially in the paper [13] by Weiss and Herman mentioned previously.

In the next paragraph, results are derived about the spatial distribution of vehicles if the temporal distribution (distribution of arrival times) is assumed to be Poisson.

Denote by S^+ the number of lines in $\mathscr{A}$ intersecting $(0, x_0)$ and $x_0 > 0$ at time zero and by S_c^+ the number of lines in $\mathscr{A}$ whose arrival times are in the interval $(-c, 0)$. We further denote by S^- the number of lines in $\mathscr{A}$ intersecting $(-x_0, 0)$, $x_0 > 0$ at time zero and by S_c^- the number of lines in $\mathscr{A}$ whose arrival times are in the interval $(0, c)$. Let us compute the quantities $Pr\{S^+ = n\}$, $Pr\{S^- = n\}$, and $Pr\{S = S^+ + S^- = n\}$. We have

$$(4.12) \qquad Pr\left\{S^+ = n\right\} = \lim_{c \to \infty} \exp\left\{-\omega c \mu_2\right\} \frac{(\omega c \mu_2)^n}{n!}.$$

where ω is the temporal density and $\mu_2 = Pr\{S^+ = 1 | S_c^+ = 1\}$. It can be shown easily that

$$(4.13) \qquad \lim_{c \to \infty} c \mu_2 = x_0 \int_0^\infty \frac{1}{v}\, dG(v).$$

We conclude that

$$(4.14) \quad Pr\left\{S^+ = n\right\} = \exp\left\{-\omega x_0 \int_0^\infty \frac{1}{v}\, dG(v)\right\}\left[\omega x_0 \int_0^\infty \frac{1}{v}\, dG(v)\right]^n (n!)^{-1}.$$

Similarly, we obtain $Pr\{S_n^- = n\} = Pr\{S^+ = n\}$ and

$$(4.15) \qquad Pr\left\{S = S^+ + S^- = n\right\} = \exp\left\{-2\omega x_0 \int_0^\infty \frac{1}{v}\, dG(v)\right\}$$

$$\cdot \left[2\omega x_0 \int_0^\infty \frac{1}{v}\, dG(v)\right]^n (n!)^{-1}.$$

We can now summarize as follows.

THEOREM 4.2. *If* $\langle t_i \rangle$ *forms a Poisson process with parameter* ω *and sequences* $\langle t_i \rangle$ *and* $\langle V_i \rangle$ *are independent, then the locations of vehicles on the highway at time* $t = 0$, *namely* $\langle x_i \rangle$, *form a Poisson process with parameter* $\omega \int_0^\infty (1/v)\, dG(v)$.

THEOREM 4.3.   *If* $\langle x_i \rangle$ *forms a Poisson process with parameter* λ^* *and sequences* $\langle x_i \rangle$ *and* $\langle V_i \rangle$ *are independent and* $\langle x_i^+ \rangle$ *denotes the positions at which the car* $K(v_0)$ *overtakes slower cars and* $\langle x_i^- \rangle$ *denotes the positions at which car* $K(v_0)$ *is overtaken by faster cars, then the two sequences* $\langle x_i^+ \rangle$ *and* $\langle x_i^- \rangle$ *form two independent (homogeneous) Poisson processes, with parameters*

$$(4.16) \qquad \lambda_+^*(v_0) = \lambda^* \int_0^{v_0} (v_0 - v) \, dG(v) \qquad \lambda_-^*(v_0) = \lambda^* \int_{v_0}^\infty (v - v_0) \, dG(v).$$

This result is analogous to the Rényi result which we developed as Theorem 4.1 except that the counting of overtakings is accomplished on the spatial axis rather than on the time axis. The next theorem provides results analogous to those in Theorem 4.2.

THEOREM 4.4.   *If* $\langle x_i \rangle$ *forms a Poisson process with parameter* λ^* *and sequences* $\langle x_i \rangle$ *and* $\langle V_i \rangle$ *are independent and the* $\langle V_i \rangle$ *are i.i.d. random variables with common distribution* $G(v) = Pr \{V \leq v\}$, *then the corresponding* $\langle t_i \rangle$ *arrival times at position* $x = 0$ *form a Poisson process with parameter* $\lambda^* E(V)$, *where* $E(V) = \int_0^\infty v \, dG(v)$.

PROOF.   The proof is again based on the binomial mixing as presented in property (B) and hence details are omitted.

5. Concluding remarks

REMARK 5.1.   On the basis of the work in the previous sections, it appears that we can view the main structural property for a nonhomogeneous Poisson field of random lines $\mathscr{L}$ in the following way. The point process obtained by the intersections of lines in the field $\mathscr{L}$ with any fixed line (p_0, θ_0) forms a Poisson process subject to the existence of the integral $\int_0^\infty h(v) \, dG(v)$ for some suitable $h(v)$, say $1/v$, $v(1 + v^2)^{-1/2}$, $(1 + v^2)^{-1/2}$, and others.

REMARK 5.2.   In light of the statement in Theorem 3.2, we can offer the more general result below for which the proof is immediate and hence is omitted.

THEOREM 5.1.   *Let* $\Theta_1, \cdots, \Theta_m$ *be m disjoint intervals on* θ *and let* $P_1, \cdots, P_m$ *be m intervals on p. Recall that* $\mathscr{L}$ *is the nonhomogeneous Poisson field defined previously, then the random variables* $N(P_i, \Theta_i)$, $i = 1, 2, \cdots, m$, *where* $N(P_i, \Theta_i) = \{no. \ of \ (p, \theta) \in \mathscr{L} \ such \ that \ p \in P_i, \ \theta \in \Theta_i\}$, $i = 1, 2, \cdots, m$, *are m independent Poisson random variables. Consequently, the lines* (p, θ) *are points of a two dimensional nonhomogeneous Poisson process with parameter* $\lambda(\Theta)$ *that depends on* θ.

Consider the strips in the (p, θ) plane where $0 \leq \theta < \frac{1}{2}\pi$ and $-\infty < p < \infty$; see Figure 2. Thus, the number of p in an interval of length I on the p axis whose θ are in the set Θ has a nonhomogeneous Poisson distribution with mean

$$(5.1) \qquad\qquad \lambda(\Theta) = \lambda I \int_\Theta dF,$$

where F is the c.d.f. on the random variable θ.

It is clear, for the homogeneous Poisson field of random lines where F is the uniform distribution, that $\int_\Theta dF$ equals the length of the interval measure for θ divided by $\frac{1}{2}\pi$. Hence, the parameter

$$(5.2) \qquad\qquad \lambda I \int_\Theta dF = \frac{2}{\pi} \lambda I \text{ (length of } \Theta),$$

and thus it depends only on the length of the set Θ and the length of I on the p axis. Thus, Theorem 5.1 holds for homogeneous Poisson fields of random lines and in this way it adds to Miles' results.

All results obtained in this paper should be capable of extension to other nonhomogeneous Poisson fields, say, where λ is the function of p, $\lambda = \lambda(p)$, or where $\lambda = \lambda(t, x)$.

REMARK 5.3. The results announced by Miles in [7] and [8] also fall out immediately from our development because there the orientations α_i are independent and uniformly, distributed, $0 \leqq \alpha_i < \pi$. Then the following result is immediate: the points of intersection of the random lines $\mathscr{A}$ and an arbitrary line (p_0, θ_0) form a Poisson process with parameter $2\lambda/\pi$.

REMARK 5.4. Based on the results stated in Theorem 3.3 (i) and Theorem 3.5, one might expect to get the following identity

$$(5.3) \qquad \lambda_\tau = \lambda_\tau \int_0^\infty \frac{(1 + v^2)^{1/2}}{v} \, dG(v) \int_0^\infty v(1 + v^2)^{-1/2} \, dG(v).$$

But the identity is true if and only if the field $\mathscr{L}$ consists of parallel lines alone. This reduces the field of random lines $\mathscr{L}$ to the case initially studied by Goudsmit [5], who employed it as a first attempt to study random lines in the plane in connection with examining the randomness of tracks left in a cloud chamber by a particle.

REMARK 5.5. The structure and properties of random lines in the plane that are developed in this paper make it possible to review and extend results in still other applications. In a paper reporting on the pattern in a planar region of one species of vegetation with respect to another, Pielou [9] defined a random pattern as one in which the alternation between species along any line transect is Markovian. In a subsequent paper, Bartlett [1] indicated that she did not establish the existence of a two state planar process that could produce this Markovian property. Switzer [11] then demonstrated the existence of a finite state random process in the plane, namely, the homogeneous Poisson field or random lines with the property that alternation among states along any straight line is Markovian. In this paper, a whole class of finite state random processes in the plane that accomplishes this is presented by our results for nonhomogeneous Poisson fields of random lines.

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